Scaling of spreading in models unidirectionally coupled to source particles

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We investigate the spreading behavior of evolving clusters using unidirectionally coupled two-level hierarchies in one spatial dimension. In the hierarchy, while only two source particles (A) hop away from each other without branching its offspring on the bottom level, different species of particles (B) evolve according to given dynamics belonging to one of known universality classes on the top level. Two levels are unidirectionally coupled from the bottom to the top level by the branching $A \to A+2B$. We derive the spreading exponent z_U of the uncoupled region of size $R_U(t) \sim t^{z_U}$ up to the first order correction in terms of the spreading exponent of source particles (z_A) and that of given dynamics of the top level (z_o) as $z_U=(1-z_A)z_o/(1-z_o)$. From the relation, z_A and z_U always satisfy the inequality $z_U \leqslant z_A$ for $z_A \geqslant z_o$. The inequality confirms that the scaling of the spreading in the slave level should follow the scaling of the source in unidirectionally coupled systems. We numerically confirm the relation for three different B-particle dynamics; annihilating random walks, branching annihilating random walks with one and two offspring which belong to the directed percolation, and the parity conserving universality class respectively.

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I. INTRODUCTION

Among nonequilibrium phase transitions, absorbing phase transitions from an active phase into an absorbing phase have been a field of growing interest during the past decades [1,2]. As in equilibrium transitions recent theoretical and numerical studies show that APT's exhibit universality and it can be classified according to conservation laws, dimensionality of systems, and symmetries of absorbing states [1–3]. However only a few universality classes have been identified so far. Directed percolation (DP) [2–5] and parity conserving (PC) [6–12] class are well-studied classes among others. While the DP class includes systems with no special attributes except the famous time reversal rapidity symmetry, the PC class includes systems obeying branching and annihilating random walks with even number of offsprings (BAWe).

As a research direction to search for further unknown universality classes, composite systems of known universality classes have been recently studied in which a system is coupled to the others in a specific manner [8,13–25]. However the coupled systems do not always exhibit critical behavior. For instance, quadratically and bidirectionally coupled DP systems still belong to the DP class despite their complex behavior [13]. Linearly and bidirectionally coupled systems of known universality classes such as DP or PC processes exhibit mean-field or nontrivial critical behavior depending on the manners of couplings [8,14–17].

Recently, linearly and unidirectionally coupled hierarchies of multispecies systems in one spatial dimension drew much attention, because they reveal critical behavior at the muticritical point where the critical points of all hierarchy levels coincide [18–25]. In general, the unidirectionally coupling between levels k and $k+1$ is defined by branching, $A_k \to A_k + nA_{k+1}$ with a positive integer n. Such a linear and unidirectional coupling, to the best of our knowledge, has been first introduced in some monomer interface growth model with the solid-on-solid (SOS) condition [18]. It turned out that the unidirectionally coupled DP dynamics from lower to higher layers is the key feature characterizing gradually decreasing critical exponents level by level of the SOS model [18]. Some polynuclear growth models [19] and models for growth of colonial organisms such as fungi and bacteria [20] also exhibit the critical behavior of the unidirectionally coupled DP processes [22]. The unidirectional coupling of PC processes was also studied in the context of interface growth via the dimer growth model [23], which also leads to the critical exponents gradually decreasing level by level at multicritical point. The coupling of models belonging to the same class only changes the scaling behavior of order parameter, and it remains the scaling of spatial and temporal correlation length, ξ and τ, unchanged [21–23]. $\xi \sim \tau^z$ with $z=\nu_\perp/\nu_\parallel$ holds near criticality [2].

Recent studies on the two-level hierarchies with the coupling of models belonging to different universality classes [24,25] suggest a simple criterion for the critical behavior of the slave (B) unidirectionally coupled to the source (A) via the coupling $A \to A+B$. According to the criterion, if the density ρ_B decays faster than ρ_A and the active region R_B spreads more slowly than R_A, then the resultant critical behavior of the slave after the coupling is completely changed. The active region (R) is usually defined as the area over which particles are distributed and it scales in time t as $R(t) \sim t^z$ at criticality [2]. In addition to the change of the order parameter exponent as in the coupling of the same class, the scaling behavior of R_B falls into the universality class of the source, i.e., $z_A = z_B$. As the spreading exponent z is defined as $z=\nu_\perp/\nu_\parallel$ [2], the change of R_B reflects the change of the scaling of both ξ and τ of the slave which scales as $\xi \sim \Delta^{-\nu_\perp}$ and $\tau \sim \Delta^{-\nu_\parallel}$ for the distance from the criticality Δ. For instance, when the source and the slave belong to the DP and PC class, respectively, it was numerically shown that the critical behavior of $R(t)$ of the slave belongs to the DP class due to the slow decay of density and fast spreading of the active region of DP systems compared to the PC systems [25].

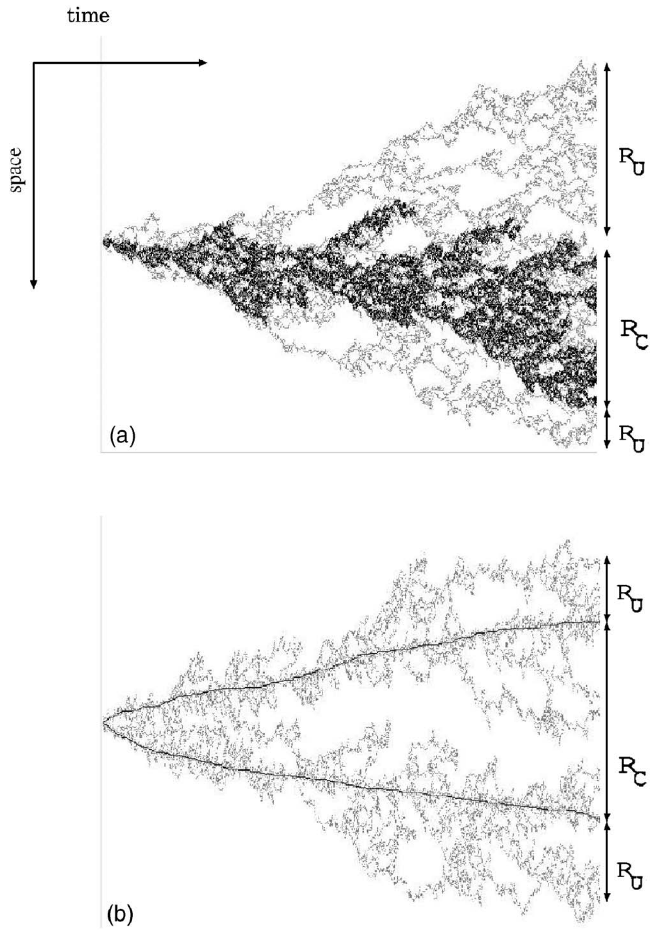

FIG. 1. Projections from above a two-level hierarchy; (a) DP-PC coupling where contact process (black) and BAW with two offsprings (gray) evolve on the bottom and the top level, respectively. (b) BAW with two offsprings on the top level (gray) is unidirectionally coupled to two source particles of the bottom level (two black lines). Each source particle algebraically spreads with the DP exponent, z_{DP}, i.e., $z_A = z_{DP}$.

In this paper, motivated by the criterion of Ref. [25], we analytically confirm the criterion for the critical behavior of the slave unidirectionally coupled to the more slowly decaying and faster spreading source using the fact that unidirectionally coupled systems exhibit the heterogeneity of the active region. The mechanism of the heterogeneity is following. In a two-level hierarchy with localized initial particles only on the source (A) level at multicritical points, a small cluster of particles is formed in slave (B) level due to the coupling $A \to A+B$. However the coupling makes the cluster of B particles grow faster than A particles do (see Fig. 1). So the inequality of the growing speed causes the cluster-size difference between the levels. As the size $R_B(t)$ of the slave level is larger than $R_A(t)$ of the source level, the source particles can affect only the region of size R_A in the slave level. The remainder region of the slave level is free from the coupling, which is called the uncoupled area of size $R_U = R_B - R_A$. At multicriticality, the three lengths scale algebraically in time as $R_B \sim t^{z_B}$, $R_A \sim t^{z_A}$, and $R_U \sim t^{z_U}$. Hence it is necessary for the proof of the above criterion to show the inequality of $z_U \leq z_A$.

For this reason, we study an efficient two-level hierarchy rather than ordinary hierarchy studied so far. In ordinary two-level hierarchy, the size R_U is the distance between the right-most particle of the slave level and that of the source level (Fig. 1). Hence the scaling of $R_U(t)$ is affected by the scaling behavior of the out-most particles of the source level. The other bulk particles of the source level can be neglected in the dynamics of $R_U(t)$. This reasoning leads us to the efficient two-level hierarchy in which only two particles reside on the source level and their spreading behavior can be fully controlled. The two particles of the source level play the role of the out-most particles of the source level in the ordinary coupled hierarchies. In the epidemic point of view, as the two biased walkers are the sources of epidemic spreading on the slave level, this simple two-level hierarchy becomes an epidemic spreading model with epidemic sources.

In the efficient model, each source particle only moves away from the other with a given spreading exponent z_A without branching its own offsprings. One can control z_A by varying one-step hopping probability $P(t)$. The two source particles create the other species of particles with a given rate on the slave level where the created species resides. The particle dynamics of the slave level can be one of any known universality classes such as the DP or PC class. The uncoupled region of the slave is now defined as the exterior of the area enclosed by the two particles of source level [Fig. 1(b)].

With this model, we derive the spreading exponent z_U of $R_U(t)$ in terms of the exponent z_A of the source particles and z_o of the given dynamics of the slave level in Sec. II. The relation clearly shows the inequality $z_U \leq z_A$ so we analytically proof the criterion. In Sec. III, the derived relation is numerically confirmed for the various dynamics of slave level such as annihilating random walks (ARW), DP, and PC dynamics. Summary and discussion are given in Sec. IV.

II. MODELS AND SCALING RELATIONS

We consider a two-level hierarchy in one dimension. On the source level, only two particles (two A's) reside and they cannot branch nor annihilate. The two particles hop only to the outward direction of each other with one-step hopping probability $P(t)$ defined as

$$P(t) = P_o t^{-\theta}. \tag{1}$$

Then the distance between two particles $R_A(t)$ is given by $R_A(t) = 2\int_{t_o}^{t} P(\tau) d\tau \sim t^{1-\theta}$ and so the spreading exponent z_A is defined as

$$z_A = 1 - \theta, \tag{2}$$

where $R_A(t) \sim t^{z_A}$. On the slave level, each particle (B) evolves with the given dynamics which may belong to the DP or the PC class and so on.

For the linear and unidirectional coupling between the two levels, each A particle creates m particles on the same and its nearest neighboring sites to the left or the right of the slave level with the unit rate as

$$A \to A + mB. \tag{3}$$

When one of the m target sites is already occupied, the branching is rejected. Starting with the empty slave level, a small cluster of B particles is formed by the coupling (3) and it grows in time t. Its size $R_B(t)$ increases algebraically at criticality as $R_B \sim t^{z_B}$ with the spreading exponent $z_B = 1/Z_B$, where Z_B is the usual dynamic exponent [1].

As the coupling of (3) makes R_B larger than R_A, the cluster of B particles is divided into two parts, namely the coupled and the uncoupled area of the size R_C and R_U as mentioned in the previous section [Fig. 1(b)]. The coupled and the uncoupled regions are defined as the interior and the exterior of the area enclosed by the two A particles, respectively. The size R_C and R_U scale with t as $R_C = R_A \sim t^{z_A}$ and $R_U \sim t^{z_U}$ at the criticality of the slave level. As $R_B = R_A + R_U \sim t^{z_A} + t^{z_U}$, the scaling of R_B is determined by the larger one, i.e., $z_B = \max[z_A, z_U]$.

Since the uncoupled region is source-free, we assume that the evolution of B particles in this region is autonomous. Then each B particle spends time $\tau_\ell \sim \ell^{1/z_o}$ to travel the distance ℓ, where z_o is the spreading exponent of the given B-particle dynamics without the coupling. To obtain the expression of z_U in terms of known z_A and z_o, we start with a very sound assumption that the right-most B particle locates at $x_B(t) = x_A(t) + R_U(t)$, where $x_A(t)$ is the position of the right A particle.

As it takes time $\tau_A = R_U(t)/v_A(t) \sim R_U(t) t^{1-z_A}$ for the A particle to reach the position $x_B(t)$, the B particle travels the distance $R_U(t+\tau_A) \sim {\tau_A}^{z_o}$ during τ_A. So we have the following relations

$$\tau_A \sim t^{1+z_U-z_A},$$

$$R_U(t+\tau_A) \sim R_U(t)^{z_o} t^{(1-z_A)z_o}. \tag{4}$$

As R_A increases algebraically ($z_A \leqslant 1$), τ_A diverges in the limit $t \to \infty$. Let us consider the ratio $X = R_U(t+\tau_A)/R_U(t)$. From the scaling of (4) and $R_U(t+\tau_A) \sim (t+\tau_A)^{z_U}$, the ratio X scales as $X \sim (1+\tau_A/t)^{z_U} \sim R_U(t)^{z_o - 1} t^{(1-z_A)z_o}$. Hence we find the scaling of $R_U(t)$ as

$$R_U(t) \sim \frac{t^{(1-z_A)z_o/(1-z_o)}}{(1+\tau_A/t)^{z_U/(1-z_o)}}. \tag{5}$$

As $\tau_A/t \sim t^{z_U - z_A} \to 0$ for $t \to \infty$ due to $z_U \leqslant z_A$, we expand the denominator in the first order of τ_A/t. Then $R_U(t)$ scales as with a constant a

$$R_U(t) \sim t^{z_U}(1 + a t^{-\Delta}), \tag{6}$$

and the exponents z_U and Δ are given by the relation

$$z_U = \frac{(1-z_A)z_o}{1-z_o} = \frac{\theta z_o}{1-z_o}, \tag{7}$$

$$\Delta = z_A - z_U = 1 - \frac{\theta}{1-z_o} \tag{8}$$

with $\theta = 1 - z_A$.

In this derivation, we find not only the leading scaling behavior of $R_U(t)$ but the first order correction to the scaling. For $z_A = z_o$ such as DP-DP or PC-PC coupling of Refs. [22–25], one finds $z_U = z_o$ and $\Delta = 0$ from Eqs. (7) and (8) as expected. When source particles spread more slowly ($z_A < z_o$) such as PC-DP coupling of Ref. [24], the scaling of the spreading on the top level should not be affected by source particles, i.e., $z_U = z_o$. However, in this case, Eqs. (7) and (8) yield $z_U > z_o$ and $\Delta < 0$ which contradict with the physical prediction. Hence Eqs. (7) and (8) are valid only for $z_A \geqslant z_o$. Finally for $z_A > z_o$ such as the DP-PC coupling of Ref. [25], the difference $z_A - z_U$ is given as $(z_A - z_o)/(1-z_o)$ so we have the inequality $z_A > z_U$. z_U also clearly continuously varies with z_A or θ. Hence the scaling of $R_U(t)$ is nonuniversal for faster spreading source particles. According to the criterion of Ref. [25] for the coupling of more slowly decaying with more quickly spreading system such as DP-PC coupling ($z_A > z_o$), the spatial and temporal correlation lengths follow the scaling of the source level. Combining three cases, we have the inequality $z_U \leqslant z_A$ from Eq. (7) so the criterion for correlation length suggested in Ref. [25] is analytically confirmed. As z_B is given by the relation $z_B = \max[z_A, z_U]$ for $z_A > z_o$, we have $z_B = z_A$ for $z_A > z_o$. On the other hand, we have $z_B = z_o$ for $z_A < z_o$. Hence when the source particles spread much faster, the spreading behavior of the slave level is completely determined by the source one.

III. MONTE CARLO SIMULATION

To confirm the relation of Eqs. (7) and (8), we perform Monte Carlo simulations for three different dynamics of ARW, BAW with odd and even offspring which belong to the DP and PC class, respectively.

The three dynamics can be integrated into following reactions. A randomly selected B particle hops to one of the nearest neighboring sites randomly with probability p. With $(1-p)$, the selected particle creates n offsprings on its n neighboring sites of the left or the right with equal probability. When two particles happen to be on the same site by hopping or branching, they immediately annihilate each other. The dynamics defines BAW with n offsprings [BAW(n)] [6,7,26].

For ARW dynamics, we only consider $p=1$ for any n. So B particles undergo only the reaction, $2B \to \varnothing$ upon colliding with each other. The spreading exponent of ARW is $z_{ARW} = 1/2$. As BAW(n) belongs to DP for odd n and PC class for even n, respectively [26], we choose $n=1$ and 2 for the dynamics of the DP and PC class [7,26]. The three dynamics give different values of z_o, but it only changes the slope of $z_o/(1-z_o)$ in $z_U - \theta$ plane.

Initially only two A particles are located on the pair of central sites of the source level, while the slave level is empty. For the coupling (3), we set $m=1$ for DP and $m=2$ for both ARW and PC dynamics to conserve the parity of the total-particle number. Then one particle is randomly selected. If the chosen particle is A, then it creates m B particles with unit probability on the same and its $m-1$ nearest neighboring sites to the left or the right with equal probability on the slave level provided that the all target sites are empty. After the branching, the A particle hops to the outward direction

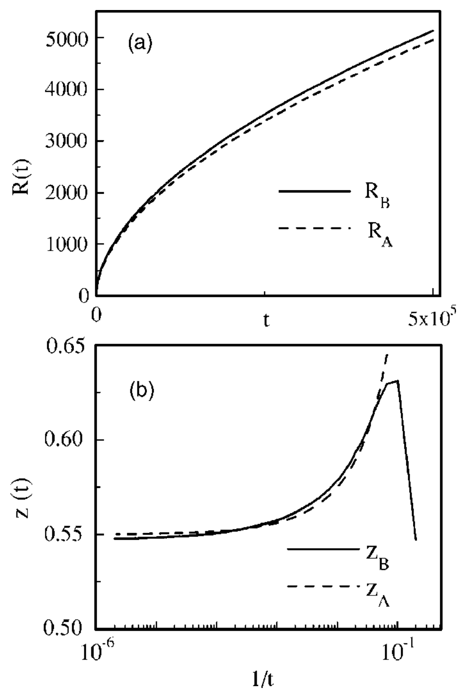

FIG. 2. ARW dynamics on the top level: (a) The spreading distance of A and B clusters, $R_A(t)$ (dashed) and $R_B(t)$ (solid) for θ=0.45. The gap $R_U(t)$ of the two regions develops and also increases algebraically. (b) Effective exponents of R_A and R_B, $z_A(t)$ (dashed) and $z_B(t)$ (solid). The two exponents coincide asymptotically.

with $P(t)$ of Eq. (1) with P_o=1. If the selected particle is B, then it performs the given dynamics without interactions with A particles.

For ARW dynamics, we measure the spreading distance $R_U(t)$ averaged over all samples for various θ values from 0.475 to 0. Figure 2(a) shows the existence of the uncoupled region of the average size $R_U(t)=\langle R_B(t)-R_A(t)\rangle$ increasing algebraically in t, where the symbol $\langle\cdots\rangle$ denotes the average over all samples. For fast spreading source particles $(z_A > z_o=\frac{1}{2})$, the spreading behavior of the slave level asymptotically follows the same scaling as that of the source level as shown in Fig. 2(b). Using the local slope of $R_U(t)$ defined as

$$z_U(t) = \ln[R_U(Mt)/R_U(t)]/\ln M, \tag{9}$$

we estimate the asymptotic value z_U for various θ values from 0.475 to 0 with M=5 (Fig. 3). Substituting $z_A=1-\theta$ and $z_o=\frac{1}{2}$ into Eq. (7), we find the relation of z_U to θ as $z_U=\theta$ for ARW dynamics. In Fig. 4(a), we plot the line $z_U=\theta$ and simulation results, which coincide with each other except the deviations by the correction (8).

To confirm the correction (8), we measure the local slope of the product $Y(t)=R_U(t)t^{\Delta-z_U}$ by multiplying $t^{\Delta-z_U}$ to Eq. (6). $Y(t)$ asymptotically scales as $Y(t)\sim bt^{\Delta}+c$, where b and c are some constants. As $Y(t)\sim bt^{\Delta}$ in the limit $t\to\infty$, it is easy to measure Δ in simulations. Figure 5 shows the local slope Δ_{eff} of $Y(t)$ for ARW dynamics similarly defined as Eq.

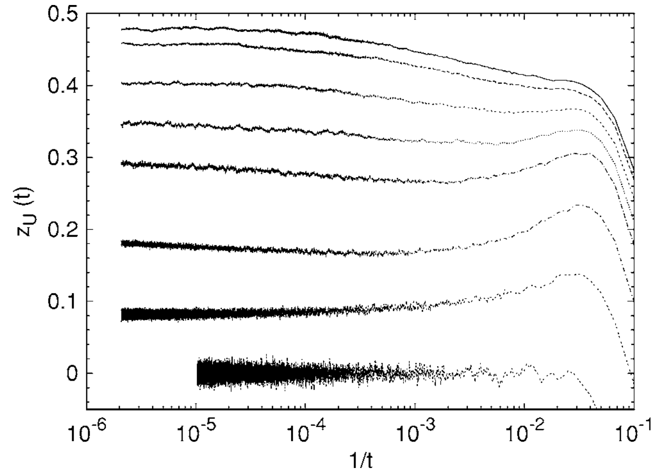

FIG. 3. ARW dynamics on the top level: Effective exponent $z_U(t)$ for θ=0.475, 0.45, 0.4, 0.35, 0.3, 0.2, 0.1, and 0 from top to bottom.

(9). Δ is given as $\Delta=1-2\theta$ from Eq. (8) for ARW dynamics and it also agrees well with our simulation results [Fig. 4(b)].

With the same method, we measure z_U and Δ for DP [BAW(1)] and PC [BAW(2)] dynamics. In the BAW(1) model, an evolving cluster in absorbing environment exhibits the power-law spreading of $R\sim t^z$ only at criticality, so we perform simulations at the criticality p_c=0.107 of the uncoupled BAW(1) model [26]. However in the BAW(2) model, the evolving cluster in vacuum exhibits the power-law spreading both at criticality and in absorbing phase. In absorbing phase, the spreading distance R follows the scaling of ARW dynamics of $R(t)\sim t^{1/2}$. Since ARW dynamics was already discussed, we measure z_U and Δ of BAW(2) at p_c=0.5105 of the uncoupled case [7].

Figures 6 and 7 show z_U and Δ for BAW(1) and BAW(2) models for various θ. As in ARW, R_U and its correction

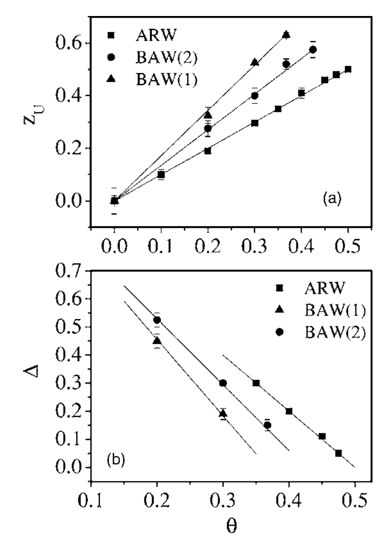

FIG. 4. The lines and symbols correspond to (a) the $z_U=S\ \theta$ line with $S=z_o/(1-z_o)$ and its numerical estimates (b) $\Delta=1-\theta/(1-z_o)$ lines and its numerical estimates for ARW, BAW(1), and BAW(2), respectively. The S and $(1-z_o)$ are 1 and 1/2 for ARW, 1.721 and 0.3675 for DP, 1.353 and 0.425 for BAW(2), respectively.

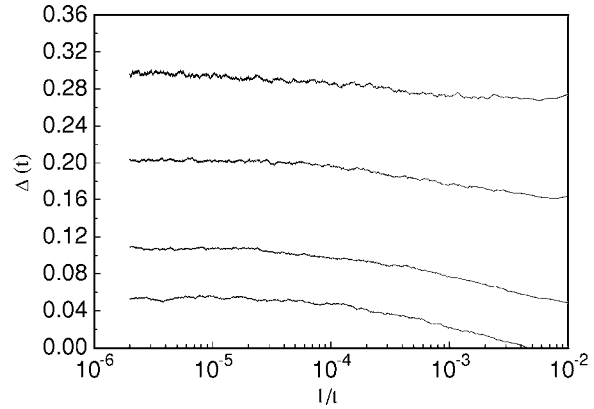

FIG. 5. ARW dynamics on the top level: Local slope Δ_{eff} of $Y(t)$ for θ=0.475, 0.45, 0.4., and 0.35 from bottom to top.

shows nonuniversal scaling depending on the scaling of source particles. For the comparison with the prediction (7), we plot numerical estimates and the line of $z_U=\theta(z_o/(1-z_o))$ with $z_o/(1-z_o)=1.721$ for DP and 1.353 for PC dynamics in Fig. 4(a). In Fig. 4(b), the line $\Delta=1-\theta/(1-z_o)$ of Eq. (8) and numerical estimates of Δ are plotted for the three dynamics. All numerical results for the three different dynamics satisfy very well with the relations (7) and (8).

The two-level hierarchy studied in Refs. [24,25] shows nontrivial scaling of R_U only for DP-PC coupling where dynamics of the bottom and the top level belong to the DP and PC class, respectively. In DP-PC coupling, as the source cluster spreads faster than that of the top level, the scaling of R_U should deviate from the PC class. From Eqs. (7) and (8), one expects z_U=0.497(5) and Δ=0.136(5) for DP-PC coupling using the value of z_{DP}=0.632 613(5) and z_{PC}=0.575(5) [5,6]. Comparing with the numerical result z_U=0.60(1) of Ref. [25], the expected value is small. However, even in our simplified model, the numerical estimate z_U=0.53(2) for $z_A=z_{DP}$ and $z_o=z_{PC}$ [Fig. 6(b)] is also larger than the expected value z_U=0.497(5) of Eq. (7) due to the correction with the small exponent Δ=0.136. Hence, the difference between the expected value of z_U from Eq. (7) for the DP-PC coupling and the numerical estimate of z_U of Ref. [25] could be explained by the correction to the scaling of $R_U(t)$ with a small exponent.

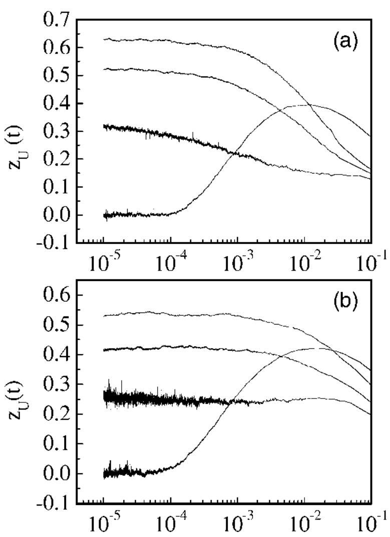

FIG. 6. $z_U(t)$: From top to bottom in each panel (a) BAW(1) for θ=0.3675, 0.3, 0.2, and 0.0. (b) BAW(2) for θ=0.425, 0.3675, 0.3, 0.2, and 0.0. θ=0.3675 and 0.425 correspond to θ_{DP} and θ_{PC}.

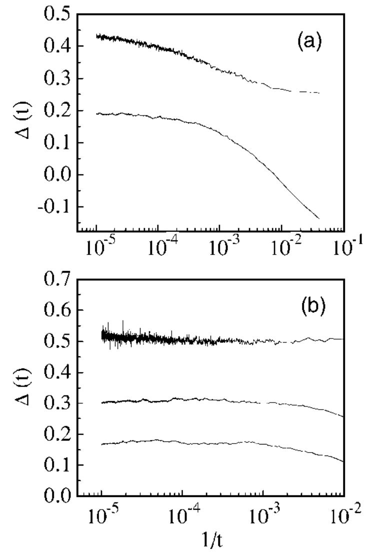

FIG. 7. $\Delta(t)$: From bottom to top in each panel (a) BAW(1) for θ=0.3, and 0.2. (b) BAW(2) for θ=0.3675, 0.3, and 0.2.

IV. SUMMARY

In summary, we investigate the scaling behavior of uncoupled regions using a simple unidirectionally coupled two-level hierarchy. On the source level, only two source particles (two A's) reside and they are driven to opposite directions without branching offsprings on their own level. On the slave level, the other species of particles (B) evolve with the given dynamics belonging to one of the known universality classes. The two levels are unidirectionally coupled from the source to the slave level via the branching $A\to A+mB$ with unit rate.

Starting with the empty slave level, a small B cluster is formed by the coupling and it spreads with the given dynamics. As the size of the B cluster is larger than the distance between two A particles due to the coupling, the B cluster can be decomposed into two parts, namely, coupled and uncoupled regions. The uncoupled region of the size $R_U(t)$ at given time t is defined as the area of the B cluster outside the region enclosed by the two A particles. Based on the assumption that B particles autonomously evolve by the given dynamics in the uncoupled region, we derive the spreading exponent z_U defined as $R_U(t)\sim t^{z_U}$ up to the first correction by heuristic arguments in terms of the spreading exponent of A particles (z_A) and that of the given B-particle dynamics (z_o).

The relation shows that z_U and the correction exponent Δ continuously vary with z_A and z_o. We numerically confirm the relations of z_U and Δ for three different B-particle dynamics; annihilating random walks, BAW(1) and BAW(2) dynamics for the DP, and the PC class, respectively. From the relation, we find that z_A and z_U always satisfy the inequality $z_U \leq z_A$. Hence we analytically confirm the criterion for the critical behavior of spatial and temporal correlations of the slave as suggested in Ref. [25]. Since our derivation of the spreading exponent z_U is only based on the given spreading behavior of each level, the relation would be valid for reaction-diffusion systems exhibiting Markovian spreading behavior without special attributes such as long-time memory effect on walks.

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